What is meant by V?

Reflections on the universe of all Sets

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Zu Inhaltsverzeichnis

CHAPTER 1

INTRODUCING *V*

1.0. PREFACE

In this chapter I'll give a brief sketch of the topic of this book. That the structure $V = \cup_\alpha V_\alpha$ is *the universe of all sets*, that the set theoretical axioms are true assertions about *V* or that a question like the *Continuum Hypothesis* is still open (since it is undecided whether this hypothesis holds or fails in *V*) are common assertions in set theory.[1] How one is to understand them, though, is not an obvious matter. My work will be concerned with such questions.

In the succeeding section 1.1. I'll specify more precisely which questions my investigations on *V* will deal with. I'll make also preliminary remarks on the kind of philosophical approach to set theoretical practice I'm going to adopt here. In sections 1.2. and 1.3. I'll summarize and reject two common ways of interpreting the philosophical status of *V*, which I'll call the *"real"-universe-problem* and the *choice-for-the-"right"-universe-problem*. Finally, in sections 1.4.-1.6., I'll introduce an alternative view of the role played by *V* in the practice of set theory. The structure *V* will be characterized as "a large place" in comparison to other contenders for the universe of all sets and the fact that *V* is appealed to as *the* universe of all sets will be understood in terms of a peculiar *epistemic attitude*, that of endorsing *the broadest possible point of view on sets*.[2]

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[1] Definitions and clarifications will follow.

[2] In this way I'm going to develop a fundamental suggestion that emerged in conversations with Hauser.

1.1. WHAT'S THE PROBLEM WITH *V*?

In the course of this introduction I'm going to reject two common claims about the philosophical status of *V*. The first view is that the attitude of appealing to *V* as to *the* universe of all sets is in need of ontological and/or epistemological justification. I'll call this view *the-"real"-universe-problem*. It is shared by people who think that, in order to possibly account for the meaningfulness of the attitude of referring to *V* as to *the* universe of all sets, one needs to give a positive answer to questions like the following. Is there a "real" universe of all sets and is it *V*? Do we possess clear intuitions for considering *V* a realization of the set theoretical axioms that is more natural than others? The "real" universe of all sets and/or our clear set theoretical intuitions would provide us with an *a priori* rationale for regarding *V* as *the* universe of all sets.

The second view I'll reject can be summed up as follows. Contemporary set theory has a methodological problem: to build a "right" model for ZFC + large cardinal axioms, i.e. a model whose content can be unfolded to a higher degree of precision than the content of *V*. Once the "right" universe has been found and it has replaced *V*, the notion of the universe of all sets – so the argument goes – will cease *ipso facto* to be problematic. This is what I'll call *choice-for-the-"right"-universe-problem*. Notice that the notion of *V* is said to be problematic here not because *V* is regarded as lacking an ontological and/or epistemological foundation but because it is viewed as a mathematically vague structure.

In what follows I'll show that none of these views of the status of the notion of *the* universe of all sets *V* is really adequate. Both just seem to me to be at odds with the practice of set theory. Consider, for instance, that in actually doing set theory one can perfectly well refer to *V* as to *the* universe of all sets and work at unfolding the content of other structures, which satisfy some of the axioms supposed to hold in *V* (i.e. ZFC + large cardinal axioms). This process is perfectly reasonable to set theorists in spite of arguments that *V* is the "real" universe or that a structure "better" than *V* must be viewed as *the* uni-

verse of all sets, i.e. considerations concerning the ontological and/or epistemological status of *V*, as well as claims about its supposed mathematical inadequacy, seem to have no influence on set theorists' attitudes towards *V* and other set theoretical universes.

So I think one should ask different questions about *V* from the ones listed above in order not to run counter to the actual practice of set theory. E.g.: how is it to be understood that *V* is appealed to as *the* universe of all sets whereas the work in many universes is inevitable in contemporary set theory and the content of some universes is even better known than the content of *V*? Were there no universe alternative to *V*, would the very notion of *the* universe of all sets be clearer? How are our views of *V* affected by what we know of other models? What difference does it make to work in *V* or, say, L, L^U, K, $V[G]$, $V^{\mathbb{B}}$ etc.? [3]

Notice that in saying that I will reject both *the "real"-universe-* and the *choice-for-the-"right"-universe-problem*, and in asking the questions as I have formulated them, I take what is being done in contemporary set theory as a sort of ultimate court of appeal for my philosophical investigations. In fact I start from the assumption that the mathematical practice of set theory exhibits a rationality of its own, which philosophy shouldn't deny but, rather, do justice to. As a result, understanding the significance of what the practitioners do and doing this in a way that doesn't contradict the evidence of the practice is a task of primary philosophical importance for me. As Kanamori puts it:

> The history and practice of mathematics in general and of set theory in particular affirms that they have achieved an *evident autonomy*, one that should resist *external* explanations, extrapolations or prescriptions. [Kanamori (1994), 474. Italics mine]

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[3] See 1.3. and 1.6. for the definitions of the structures mentioned here. A detailed description of V, L, L^U, K will be given in Chapter 2, while $V[G]$ and $V^{\mathbb{B}}$ will be extensively described in Chapter 4. This chapter presupposes some familiarity with the structures mentioned.

Claiming, as the supporters of *the-"real"-universe-problem* do, that set theorists' views of *V* are in need of an *a-priori* foundation just means denying the "evident autonomy" of set theory. Claiming that set theorists have to find a model that yields the "right" solutions to open set theoretical questions and replace *V* with it, means calling for a revision of the practice of set theory. This is at odds with the fact that, as I'll show, one cannot really get rid of *V* in one's work with sets.

In conclusion, I think that what an analysis of *V* should aim at, is just to describe what mathematicians mean and what expectations they have when they refer to *V* as to *the* universe of all sets. Anyway set theorists' attitudes towards *V* are not so obvious a matter that one can simply report them as observation data. They have to be made explicit through a fine interpretation of what is implicit in the practice of set theory. Of course I don't deny that better interpretations of *V* and its significance than the ones I'm going to sketch in this book can be given.

1.2. THE-"REAL"-UNIVERSE-PROBLEM

My understanding of *V* won't result in a characterization of this structure as a privileged realization of the set theoretical axioms from an ontological or an epistemological point of view. I.e. in what follows *V* will be interpreted neither as a picture of "some well determined reality in which Cantor's conjecture must be true or false" nor as a model for ZFC + large cardinal axioms that is directly suggested by set theoretical intuitions that "force themselves upon us as being true".[4]

Indeed it is often observed by set theorists, in particular with regard to the adoption of new axioms, that there is no need to appeal to *a-priori* motivations that explain why something should be as it is *de facto*, in order to account for set theorists' attitudes. Martin, for instance, justifies his decision not to deal with ontological matters in his (1998) as follows:

[4] Gödel (1964), 476-484.